§. Разложение Фурье рядов Эйзенштейна.

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!} \qquad B_1 = \frac{1}{6} \quad B_2 = \frac{1}{30} \dots$$

Утв. $k>1$ $\zeta(2k) = \frac{2^{2k}}{(2k)!} B_k \pi^{2k}$

Д-во: $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

$$z \operatorname{ctg} z = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = iz + \frac{2iz}{e^{2iz}-1} \qquad x = 2iz$$

$$z \operatorname{ctg} z = 1 - \sum_{k=1}^{\infty} B_k \frac{2^{2k} z^{2k}}{(2k)!}$$

$$\sin z = z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

(*) $$z \operatorname{ctg} z = 1 - \sum_{n=1}^{\infty} \frac{2z^2/n^2\pi^2}{1 - z^2/n^2\pi^2} = 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{2k}}{n^{2k}\pi^{2k}} =$$

$$= 1 - 2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} n^{-2k}\right)\frac{z^{2k}}{\pi^{2k}} \qquad \boxed{\frac{2\zeta(2k)}{\pi^{2k}} = \frac{B_k 2^{2k}}{(2k)!}}$$

$$\sigma_k(n) = \sum_{d|n} d^k \qquad q = e^{2\pi i z}$$

Утвержд. $k \geq 2$ четное
$$\boxed{G_k(z) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n}$$

$$z \operatorname{ctg} z = 1 + 2\sum \frac{z^2}{z^2 - n^2\pi^2}$$

$$\pi \operatorname{ctg} \pi z = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2-n^2} = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right)$$

$$\operatorname{ctg} z = \frac{i}{e^{2iz}-1} + i \qquad \pi \operatorname{ctg}(\pi z) = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i \sum_{n=1}^{\infty} q^n$$

$$\sum \frac{1}{(n+z)^k} = \frac{1}{(k-1)!}(-2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n$$

$$G_{2k}(z) = \sum_{m,n}{}^{*} \frac{1}{(nz+m)^{2k}} = 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(nz+m)^{2k}} =$$

$$2\zeta(2k) + 2\frac{(-2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sum_{a=1}^{\infty} a^{2k-1} q^{an} = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n) q^n$$

§. Размерности пр-в модулярных форм.

$\Gamma = \Gamma(1)$

$\nu_p(f)$ – порядок f в точке p.

$\nu_p(f) = \nu_{\gamma p}(f)$

Теорема. $f \in M_k(\Gamma)$ $f \neq 0$. Тогда

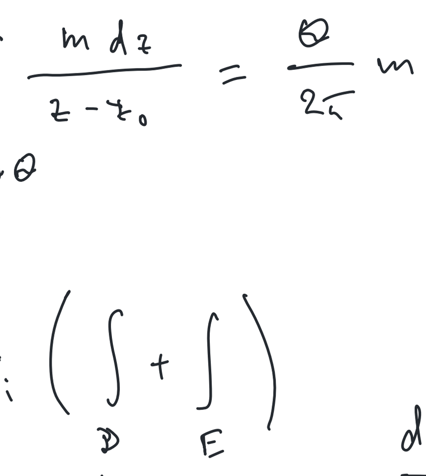
$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) + \sum_{\substack{p\in\Gamma\backslash H \\ p \neq i,\rho}} \nu_p(f) = \frac{k}{12}$$

Д-во $\frac{f'}{f}$ $\qquad f = (z-z_0)^m + \dots$

$$\frac{f'}{f} = \frac{m}{z-z_0} + \dots$$

ф-ла Коши $\sum$ нулей $= \frac{1}{2\pi i}\int_{C} \frac{f'(z)}{f(z)} dz$

$f(z) = f(z+1)$



$$\frac{1}{2\pi i}\int_{ABCDE} \frac{f'}{f} dz \qquad q = e^{2\pi i z} \qquad \tilde{f}(q) = \sum a_n q^n$$

$$\int_B \frac{df}{f} = \int_{\circlearrowleft} \frac{d\tilde{f}}{\tilde{f}} = -\nu_\infty(f) \qquad e^{-2\pi a}$$

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{z_0 \varepsilon} \frac{m\,dz}{z-z_0} = \frac{\theta}{2\pi} m$$

В точке i: $\int = \frac{1}{2}\nu_i(f)$

ρ: $\int = \frac{1}{3}\nu_\rho(f)$

4) $\frac{1}{2\pi i}\left(\int_D + \int_E\right) = I$

$$f(Sz) = z^k f(z)$$

$$\frac{df(Sz)}{f(Sz)} = \frac{d(z^k f(z))}{z^k f(z)} = k\frac{dz}{z} + \frac{df(z)}{f(z)}$$

$$d(z^k f(z)) = kz^{k-1}f(z) + z^k df(z)$$

$$I = \frac{1}{2\pi i}\int_D \left(\frac{df(z)}{f(z)} + k\frac{dz}{z} - \frac{df(z)}{f(z)}\right) = \frac{k}{12}$$

Следствие k – четное $\Gamma = \Gamma(1)$ $\qquad \nu_\infty + \frac{1}{2}\nu_i + \frac{1}{3}\nu_\rho + \sum \nu_p = \frac{k}{12}$

a) $M_k = 0$ $k<0$ или $k=2$;

b) $M_k = \mathbb{C}$ $k=0$;

c) $\dim_{\mathbb{C}} M_k = 1$ $k = 4, 6, 8, 10, 14$. $M_k = \mathbb{C}E_k$

d) $S_k = 0$ $k<12$ и $k=14$

при $k>14$ $S_k = \Delta M_{k-12}$

e) $M_k = S_k \oplus \mathbb{C}E_k$ $k>2$

$f \in M_0$

$f(z_0) = c \neq 0$. $\qquad f(z) - c \in M_0$

$f(z_0) - c = 0$

d) $f \neq 0 \in S_k$ $k \geq 12$.

$$\Delta = g_2^3 - 27g_3^2$$

$$\forall f \in S_k \quad \frac{f}{\Delta} \in M_{k-12}$$

$f \in M_k$ $\qquad E_k(\infty) = 1 \neq 0$

$M_{k-12} \ni g\Delta = (f - cE_k)(\infty) = 0 \in S_k(\Gamma)$

$$E_4 E_6 = E_{10}$$

Следствие $f \in \mathbb{C}[E_4, E_6]$.

§. Немного о j-инварианта.

$$j(z) := \frac{1728 g_2^3}{\Delta} = 1728 \frac{E_4^3}{E_4^3 - E_6^2}$$

Утв. j реализует биекцию между $\Gamma\backslash H$ и S^2.

$\Delta(\infty) = 0$ $\quad g_2(\infty) \neq 0$. $\Rightarrow$ j – простой полюс в ∞

$$\frac{1728 g_2^3 - c\Delta}{\Delta} \in M_{12} \qquad j(z) - c$$

Следствие $\forall$ модул. ф-ция есть рац. ф-ция от $j(z)$.

f – мод. ф-ция $\quad z_1, z_2 \dots z_k$

$$g(z) = f(z)\prod_{n=1}^{k}\left(j(z) - j(z_n)\right)$$

$g\Delta^k \in M_{12k}$ – многочлен от E_4 и E_6

$$\frac{E_6^2}{\Delta} \qquad \frac{E_4^3}{\Delta} = j \qquad \frac{E_4^p E_6^r}{\Delta^k} \qquad k = 4p + 6r$$

$$\frac{1728}{j} = \frac{E_4^3 - E_6^2}{E_4^3} = 1 - \frac{E_6^2/\Delta}{E_4^3/\Delta}$$